

# A NOTE ON THE NUMBER OF $S$ -DIOPHANTINE QUADRUPLES

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ABSTRACT. Let  $(a_1, \dots, a_m)$  be an  $m$ -tuple of positive, pairwise distinct integers. If for all  $1 \leq i < j \leq m$  the prime divisors of  $a_i a_j + 1$  come from the same fixed set  $S$ , then we call the  $m$ -tuple  $S$ -Diophantine. In this note we estimate the number of  $S$ -Diophantine quadruples in terms of  $|S| = r$ .

## 1. INTRODUCTION

There is a vast amount of papers concerning the problem of determining the number of prime divisors of products of the form

$$\prod_{a \in A, b \in B} (a + b) \quad \text{and} \quad \prod_{a \in A, b \in B} (ab + 1),$$

where  $A$  and  $B$  are finite sets of positive integers. In particular, the first product has been studied, first by Erdős and Turán [4] and their investigations were continued in a series of papers by Sárközy and Stewart (see e.g. [12, 13]). The second product was studied e.g. by Győry, Sárközy and Stewart [8], Sárközy and Stewart [14], and others.

In their paper [8], Győry, Sárközy and Stewart conjectured that the largest prime factor of

$$(ab + 1)(ac + 1)(bc + 1), \quad 0 < a < b < c$$

goes to infinity as  $c$  does. This conjecture has been proved by Corvaja and Zannier [3] and Hernandez and Luca [9], independently. Due to the application of the Subspace theorem their results stay ineffective. The best approach to estimate the growth rate of the largest prime factor of  $(ab + 1)(ac + 1)(bc + 1)$  is due to Luca [10], who proved that for every fixed finite set of primes  $S$ , there exist ineffective constants  $C_S$  and  $C'_S$  such that

$$((bc + 1)(ac + 1))_{\bar{S}} > \exp \left( C_S \frac{\log c}{\log \log c} \right)$$

whenever  $a < b < c$  with  $c > C'_S$ , where  $(\cdot)_{\bar{S}}$  denotes the  $S$ -free part.

In case of quadruples effective results are known. For example, Stewart and Tijdeman [15], proved that the largest prime factor of

$$\prod_{\substack{a, b \in A, \\ a \neq b}} (ab + 1)$$

with  $|A| \geq 4$ , is at least  $C \log \log \max A$ , where  $C$  is an effective computable constant.

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Let  $S$  be a fixed, finite set of primes. In view of classical Diophantine  $m$ -tuples we call an  $m$ -tuple  $(a_1, \dots, a_m)$  of positive, pairwise distinct, integers  $S$ -Diophantine if for all  $1 \leq i < j \leq m$  the set of prime divisors of  $a_i a_j + 1$  is contained in  $S$ . The results of Corvaja, Zannier [3] and Hernandez, Luca [9] yield the finiteness of  $S$ -Diophantine triples for fixed  $S$ . Although we are able to estimate the number of  $S$ -Diophantine triples due to a result of Bugeaud and Luca [2], it is in principle not possible to determine all triples with the methods currently available.

In contrast to the case of triples we can, in principle, effectively determine all  $S$ -Diophantine quadruples for a given set  $S$  due to the result of Stewart and Tijdeman [15]. Recently, Szalay and Ziegler [16], established an efficient algorithm to determine all  $S$ -Diophantine quadruples for a given set  $S$  of primes, provided  $|S| = 2$ . In particular, the results of Szalay and Ziegler [17, 18, 16], suggest that for  $|S| = 2$  no quadruple exists at all.

The aim of this note is to give upper bounds for the number of  $S$ -Diophantine quadruples for fixed sets  $S$  of  $r$  primes. We need the following notations. Let  $\Gamma$  be a multiplicative subgroup of  $\mathbb{Q}^*$  of rank  $r$  and denote by  $A(n, r)$  an upper bound for the number of non-degenerate solutions  $(x_1, \dots, x_n) \in \Gamma^n$  to the linear  $S$ -unit equation

$$(1) \quad a_1 x_1 + \dots + a_n x_n = 1, \quad a_i \in \mathbb{Q}^*.$$

We call a solution to (1) non-degenerate if no subsum on the left hand side of equation (1) vanishes. With this notation at hand our main result is:

**Theorem 1.** *Let  $S$  be a set of  $r$  primes. Then there exist at most*

$$(A(5, r) + A(2, r)^2)A(3, r)$$

*$S$ -Diophantine quadruples. If  $r = 2$  or  $2 \notin S$ , then there exist at most*

$$A(5, r)A(3, r)$$

*$S$ -Diophantine quadruples.*

Using the best estimates for  $A(n, r)$  currently available we obtain

**Corollary 1.** *Let  $S$  be a set of  $r$  primes. Then there exist at most*

$$\exp(27398 + 5126r)$$

*$S$ -Diophantine quadruples.*

In the next section we prove Theorem 1 and in the third section we discuss the number of solutions to the  $S$ -unit equation (1) and establish Corollary 1.

## 2. A SYSTEM OF $S$ -UNIT EQUATIONS

Assume that  $(a, b, c, d)$  is an  $S$ -Diophantine quadruple, with  $a < b < c < d$ . We write,

$$\begin{aligned} ab + 1 &= s_1, & ac + 1 &= s_2, & ad + 1 &= s_3, \\ bc + 1 &= s_4, & bd + 1 &= s_5, & cd + 1 &= s_6. \end{aligned}$$

With these notations we have

$$\begin{aligned} abcd &= s_1 s_6 - s_1 - s_6 + 1 \\ &= s_2 s_5 - s_2 - s_5 + 1 \\ &= s_3 s_4 - s_3 - s_4 + 1 \end{aligned}$$

and obtain the following system of  $S$ -unit equations

$$(2) \quad \begin{aligned} s_1 s_6 - s_1 - s_6 - s_2 s_5 + s_2 + s_5 &= 0, \\ s_1 s_6 - s_1 - s_6 - s_3 s_4 + s_3 + s_4 &= 0. \end{aligned}$$

Let us consider the first equation more closely and write  $y_1 = s_1 s_6$ ,  $y_2 = s_1$ ,  $y_3 = s_6$ ,  $y_4 = s_2 s_5$ ,  $y_5 = s_2$  and  $y_6 = s_5$ . Then the first equation of system (2) takes the form

$$y_1 - y_2 - y_3 - y_4 + y_5 + y_6 = 0$$

and has at most  $A(5, r)$  projective solutions in  $\mathbb{P}^5(\Gamma)$  such that no subsum vanishes, where  $\Gamma \subset \mathbb{Q}^*$  is the multiplicative group generated by  $S$ . Note that each projective solution yields at most one solution  $(s_1, s_2, s_5, s_6)$ . Indeed, assume  $(s_1, s_2, s_5, s_6)$  and  $(s'_1, s'_2, s'_5, s'_6)$  correspond to the same projective solution. Then there is a rational number  $\rho \neq 0$  such that  $s_1 = \rho s'_1$ ,  $s_6 = \rho s'_6$ ,  $s_2 = \rho s'_2$ ,  $s_5 = \rho s'_5$  and  $s_1 s_6 = \rho s'_1 s'_6$ . But this implies that  $s_1 s_6 = \rho^2 s'_1 s'_6 = \rho s'_1 s'_6$ , thus  $\rho = 1$  and  $s_i = s'_i$  for  $i = 1, 2, 5, 6$ .

So we are left to count how many solutions exist with vanishing subsums. Of course there exist no vanishing one-term subsums. Two-term vanishing subsums imply either

- $s_i = s_j$  for  $i \neq j$  which is impossible, unless  $i, j \in \{3, 4\}$  which is excluded, or
- $s_i = s_1 s_6 > abcd > cd + 1 \geq s_6 \geq s_i$  for some  $i \in \{1, 2, 5, 6\}$  which is also a contradiction, or
- $s_i = s_2 s_5 > abcd > cd + 1 \geq s_6 \geq s_i$  for some  $i \in \{1, 2, 5, 6\}$  which is also a contradiction, or
- $s_1 s_6 = s_2 s_5$ , which implies  $ab + cd + 2 = s_1 + s_6 = s_2 + s_5 = ac + bd + 2$ ; hence,  $(c - b)(d - a) = 0$ ; i.e.,  $d = a$  or  $b = c$ , again a contradiction.

Therefore no two-term subsums vanish. Since four- and five-term vanishing subsums imply the existence of two- and one-term vanishing subsums, respectively, we are left with the case of three-term vanishing subsums.

Without loss of generality we may assume that the vanishing three-term subsum contains  $s_1 s_6$ . Thus we distinguish whether  $s_2 s_5$  is contained in the vanishing subsum or not. Let us consider the case that  $s_2 s_5$  is not contained. Then we have an equation of the form  $s_1 s_6 = \pm s_i \pm s_j$ . Since  $s_1 = ab + 1 > 2 \cdot 1 + 1 > 2$  we have  $s_1 s_6 > 2s_6 > s_i + s_j$  and this case yields no solution.

Therefore both  $s_1 s_6$  and  $s_2 s_5$  are contained in the same vanishing three-term subsum and we are left with four systems of  $S$ -unit equations namely

$$(3) \quad \begin{aligned} s_1 s_6 - s_5 s_2 &= s_1, & \text{and} & & s_6 &= s_5 + s_2 \\ s_1 s_6 - s_5 s_2 &= s_6, & \text{and} & & s_1 &= s_5 + s_2 \\ s_1 s_6 - s_5 s_2 &= -s_2, & \text{and} & & s_1 + s_6 &= s_5 \\ s_1 s_6 - s_5 s_2 &= -s_5, & \text{and} & & s_1 + s_6 &= -s_2. \end{aligned}$$

Note that only the first equation of (3) is possible since by assumption  $s_1 < s_2 < s_5 < s_6$ . Let  $y_1 = s_1 s_6$ ,  $y_2 = s_5 s_2$  and  $y_3 = s_1$ . Then the  $S$ -unit equation

$$y_1 - y_2 = y_3$$

has at most  $A(2, r)$  projective solutions  $(y_1, y_2, y_3) \in \mathbb{P}^2(\Gamma)$ . Note that all solutions that yield  $S$ -Diophantine quadruples are non-degenerate, since a vanishing subsum would imply either  $s_1 s_6 = 0$  or  $s_2 s_5 = 0$  or  $s_1 = 0$ . Each projective solution yields

only one possibility for  $s_6$ . Indeed, assume that  $(s_1, s_2, s_5, s_6)$  and  $(s'_1, s'_2, s'_5, s'_6)$  yield the same projective solution. Then there exists  $\rho \in \mathbb{Q}^*$  such that  $s_1 s_6 = \rho s'_1 s'_6 = s_1 s'_6$ , since  $s_1 = \rho s'_1$ , i.e.  $s_6 = s'_6$ . We have now at most  $A(2, r)$  possible values for  $s_6$ ; i.e., we are reduced to at most  $A(2, r)$  equations of the form

$$a = s_5 + s_2$$

with  $a = s_6 \neq 0$  fixed. Thus, system (3) yields at most  $A(2, r)^2$  solutions.

In view of the second statement of Theorem 1 we note that any equation of system (3) cannot have a solution if  $2 \notin S$ . Otherwise  $s_6$  is odd but  $s_5 + s_2$  would be even. In case of  $r = 2$ , this implies  $S = \{2, p\}$  and the equation  $s_6 = s_5 + s_2$  turns into

$$(4) \quad 2^{\alpha_6} p^{\beta_6} = 2^{\alpha_5} p^{\beta_5} + 2^{\alpha_2} p^{\beta_2}.$$

Considering 2-adic and  $p$ -adic valuations, equation (4) reduces to the Diophantine equation

$$2^x - p^y = \pm 1.$$

By Mihăilescu's solution of Catalan's equation [11], only  $p = 3$  is possible. On the other hand, Szalay and Ziegler [16] showed that no  $\{2, 3\}$ -Diophantine quadruple exists.

Altogether, we have proved the following result.

**Lemma 1.** *The first  $S$ -unit equation in (2) has at most  $A(5, r) + A(2, r)^2$  solutions. If  $r = 2$  or  $2 \notin S$ , then there exist at most  $A(5, r)$  solutions.*

Now, we turn to the second equation of system (2). By Lemma 1, the first equation in (2) yields at most  $A(5, r) + A(2, r)^2$  or  $A(5, r)$  many possibilities for the pair  $(s_1, s_6)$  respectively. Thus, we assume that the second equation of system (2) is of the form

$$(5) \quad s_3 s_4 - s_3 - s_4 = a \quad \text{with } a \in \mathbb{Q} \text{ fixed.}$$

But  $S$ -unit equation (5) has at most  $A(3, r)$  solutions provided  $a \neq 0$ . Indeed no degenerate solution exists since a vanishing subsum on the left side of equation (5) would imply either

- $s_3 s_4 = s_3$  and therefore  $s_4 = 1$ , or
- $s_3 s_4 = s_4$  and therefore  $s_3 = 1$ , or
- $s_3 + s_4 = 0$  and therefore  $s_3 s_4 < 0$ .

Let us note that  $a = s_6 s_1 - s_6 - s_1 > 2s_6 - s_6 - s_1 > 0$ , and therefore we have proved the following lemma.

**Lemma 2.** *The Diophantine system (2) has at most  $(A(5, r) + A(2, r)^2)A(3, r)$  solutions. If  $r = 2$  or  $2 \notin S$ , then there exist at most  $A(5, r)A(3, r)$  solutions.*

In order to prove Theorem 1 it remains to prove that for fixed integers  $s_1, \dots, s_6$  there exists at most one quadruple  $(a, b, c, d)$ . Since

$$\begin{aligned} a &= \sqrt{\frac{(s_1 - 1)(s_2 - 1)}{s_4 - 1}}, & b &= \sqrt{\frac{(s_1 - 1)(s_4 - 1)}{s_2 - 1}}, \\ c &= \sqrt{\frac{(s_2 - 1)(s_4 - 1)}{s_1 - 1}}, & d &= \sqrt{\frac{(s_5 - 1)(s_6 - 1)}{s_4 - 1}}, \end{aligned}$$

the proof of Theorem 1 is complete.

## 3. PROOF OF COROLLARY 1

A look through the vast literature on  $S$ -unit equations shows that for  $S$ -unit equations over the rationals the best result is due to Evertse [5] provided  $|S| = 2$  and due to Amoroso and Viada [1] in the general case. Therefore we may assume  $A(2, r) = 3 \cdot 7^{3+2r}$  and  $A(n, r) = (8n)^{4n^4(n+r+1)}$ . A look at the proof of the bound for  $A(n, r)$  in [1] shows that this bound is derived by the recursive relation

$$A(n, r) \leq 2^n A(n-1, r) B(n, r+1),$$

where  $B(n, r) = (8n)^{6n^3(n+r)}$ . Note that this recursive estimate already appears in [7]. However, recursively computing  $A(n, r)$  we obtain

$$A(3, r) \leq 8 \cdot 3 \cdot 7^{3+2r} \cdot 24^{162(4+r)} < \exp(2069 + 518.8r).$$

Continuing these computations we arrive at

$$A(5, r) < \exp(25329 + 4616.3r).$$

With these numbers plugged into Theorem 1, we obtain Corollary 1.

*Remark 1.* Let us note that directly applying the bounds due to Evertse [5] and Amoroso and Viada [1] would yield the slightly worse bound  $\exp(73801 + 15378r)$  for the number of  $S$ -Diophantine quadruples. A closer inspection of the computation of the quantity  $B(n, r)$  due to Amoroso and Viada [1] and Evertse et.al. [7] would further improve the bounds also in view of the new improvements of the Subspace Theorem due to Evertse and Ferretti [6]. But we are afraid that the gain is too small for such an effort.

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